ON THE PLASTICITY LIMIT OF COMPOSITE MATERIALS

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The problem of determining the plasticity limit of a composite medium is considered. Utilizing the variational principles of ideal plasticity theory [1], a dissipative function of the macromedium is found, which contains rigidly plastic components. Problems of this kind were investigated for the case of an elastic medium in [2, 3].

1. Let components of an ideal rigidly plastic medium with different plasticity limits fill an infinite Euclidean space at each point x_i. The components of the medium are strongly connected and satisfy the Mises plasticity condition

$$s_{ij}s_{ij} = (k^\circ + k')^2 \tag{1.1}$$

where s_{ij} is the stress deviator, k° the average plasticity limit with respect to the volume, k' the fluctuations in the plasticity limit which are assumed to be homogeneous random functions.

Let us apply the variational principles of plasticity theory to calculate the dissipative function of the macromedium. On the boundary of some volume V let a velocity field v_i° be given which corresponds to the homogeneous state of strain ε_{ij}°. The actual field of strain rates satisfying the prescribed boundary conditions corresponds to the minimum of the functional [1]

$$D = \frac{1}{V}\int_V (k^\circ + k')\sqrt{\varepsilon_{ij}\varepsilon_{ij}}\, dV \tag{1.2}$$

For a sufficiently large volume V the integral (1.2) can be considered the dissipation density which corresponds to the dissipative function of the macromedium $D(\varepsilon_{ij}^\circ)$. Let us take $\varepsilon_{ij} = \varepsilon_{ij}^\circ$ in (1.2) (this corresponds to the Voigt average in the case of an elastic medium), then we obtain the upper bound for the dissipative function from (1.2)

$$D(\varepsilon_{ij}^\circ) < k^\circ \sqrt{\varepsilon_{ij}^\circ\varepsilon_{ij}^\circ} \tag{1.3}$$

We obtain as a corollary of the second theorem on ultimate equilibrium [1] that the flow surface of the macromedium corresponding to the dissipative function $D(\varepsilon_{ij}^\circ)$ lies within the Mises flow surface $s_{ij}^\circ s_{ij}^\circ = k^{\circ 2}$.

Let us examine the average for a constant stress field $\sigma_{ij} = \sigma_{ij}^\circ$ (in the case of an elastic medium this corresponds to the Reuss average), then we obtain from the first ultimate equilibrium theorem that the statically safe field σ_{ij}° should satisfy the condition $s_{ij}^\circ s_{ij}^\circ = k^*$, where k^* is the least plasticity limit of the medium components.

Therefore, the flow surface of the macromedium lies between Mises cylinders of radii k° and k^*.

Let us utilize the condition of minimum of the functional (1.2) to calculate the dissipative function $D(\varepsilon_{ij}^\circ)$ approximately.

Let us represent the integrand in (1.2) as a Taylor series in the fluctuations k', ε_{ij}° ($\varepsilon_{ij} = \varepsilon_{ij}^\circ + \varepsilon_{ij}'$), and let integration extend over the infinite space.

By virtue of the homogeneity of the random functions and the boundary conditions, the average with respect to the volume and the mathematical expectation will agree.

The relation (1.2) may now be represented as the series

$$D(\varepsilon_{ij}^\circ) = k^\circ J_0 + \langle k'\varepsilon_{ij}'\rangle \frac{\varepsilon_{ij}^\circ}{J_0} + \frac{k^\circ}{2J_0}\left(\delta_{im}\delta_{jn} - \frac{\varepsilon_{ij}^\circ\varepsilon_{mn}^\circ}{J_0^2}\right)\langle\varepsilon_{ij}'\varepsilon_{mn}'\rangle + \cdots$$

$$J_0 = \sqrt{\varepsilon_{ij}^\circ\varepsilon_{ij}^\circ} \tag{1.4}$$

where the angular brackets denote the average.

All the moments of the random fluctuations not above second order are written down in (1.4). The fluctuations are assumed sufficiently small so as to guarantee convergence of the series (1.4).

2. Let us assume that the fluctuations in the stress deviator can be represented approximately as

$$s_{ij}' = \eta_0\varepsilon_{ij}' + \eta_{ij}k' \tag{2.1}$$

where the quantities η_0, η_{ij} must ne constant by virtue of the homogeneity of the random functions under consideration. These constants are determined from the condition of minimum of the functional (1.4).

Since the mean stresses are constant, the fluctuations s'_{ij} must satisfy the equilibrium equations

$$\sigma'_{,i} + \eta_0\varepsilon'_{ij,j} + \eta_{ij}k'_{,j} = 0 \tag{2.2}$$

Let us append the compatibility and incompressibility equations

$$\varepsilon'_{ij} = {}^1\!/_2\,(v'_{i,j} + v'_{j,i}) \qquad \varepsilon'_{ii} = 0 \tag{2.3}$$

where σ', v'_i are the fluctuations in the hydrostatic pressure and the velocity field.

We obtain the solution of the system of equations (2.2), (2.3) by using Fourier transforms, where k_i denotes the transformation parameters in the three variables x_i. The solution of (2.3), (2.2) is of the form

$$\eta_0\varepsilon_{ij}' = \int_{-\infty}^{\infty} [2k_ik_jk_kk_m\eta_{km} - m(k_ik_k\eta_{jk} + k_jk_k\eta_{ik}]\frac{\varkappa}{m^2}e^{ik_nx_n}\,dk \quad (m = k_ik_i) \tag{2.4}$$

where $\varkappa'$ is a function of the variables k_i defining the spectral decomposition of the random function

$$k' = \int_{-\infty}^{\infty} \varkappa' e^{ik_nx_n}\,dk \tag{2.5}$$

Integration is over the whole space of the variables k_i.

Let us assume the random function k' to be isotropic, then [4]

$$\langle\varkappa'(k)\,\varkappa'(k')\rangle = \Lambda(m)\,\delta(k_i - k_i') \tag{2.6}$$

where $\Lambda(m)$ is the spectral density of the function k'.

Since the δ-function from (2.4) and (2.5) enters into (2.6), we obtain

$$\langle k'\varepsilon_{ij}'\rangle = \frac{1}{\eta_0}\int_{-\infty}^{\infty}\left(2\frac{k_ik_jk_kk_m}{m}\eta_{km} - k_ik_k\eta_{jk} - k_jk_k\eta_{ik}\right)\frac{\Lambda(m)}{m}\,dk \tag{2.7}$$

Evaluation of integrals of the form

$$A_{ijkmnpqr} = \int_{-\infty}^{\infty} k_ik_jk_kk_mk_nk_pk_qk_r\frac{\Lambda(m)}{m^4}\,dk \tag{2.8}$$

is required in the sequel.

Since $\Lambda(m)$ is an isotropic function, the integral under consideration is an isotropic tensor symmetric in all subscripts.

An isotropic tensor with constant components can be represented as a linear combination of products of the tensor δ_{ij}. After symmetrization with respect to all subscripts, we find that the tensors under consideration are proportional to the following:

$$\begin{gathered}\delta_{ijkm} = \delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk} \\ \delta_{ijkmnp} = \delta_{ij}\delta_{kmnp} + \delta_{ik}\delta_{jmnp} + \delta_{in}\delta_{jknp} + \delta_{in}\delta_{jkmp} + \delta_{ip}\delta_{jkmn} \\ \delta_{ijkmnpqr} = \delta_{ij}\delta_{kmnpqr} + \delta_{ik}\delta_{jmnpqr}\end{gathered} \qquad (2.9)$$

The integral (2. 8) can now be represented as

$$A_{ijkmnpqr} = A\delta_{ijkmnpqr} \qquad (2.10)$$

Integrals for the form (2. 8) which contain a lesser number of pairwise products of the vectors k_i/m, can be evaluated by means of a convolution of corresponding subscripts in (2. 10). The relations

$$\delta_{ijkmnpqq} = 9\delta_{ijkmnp}, \quad \delta_{ijkmnn} = 7\delta_{ijkm}, \quad \delta_{ijkk} = 5\delta_{ij} \qquad (2.11)$$

should hence be utilized.

Taking into account that [4] $\int\limits_{-\infty}^{\infty} \Lambda(m)\, dk = d^2$

where d is the variance of the random function k', we obtain from (2. 8), (2. 10) and (2. 11)

$$9 \cdot 7 \cdot 5 \cdot 3A = d^2 \qquad (2.12)$$

Utilizing the properties of the tensors (2. 9), we obtain the following identities:

$$\begin{gathered}\delta_{ikmp}\varepsilon_{ij}^{\circ}\varepsilon_{mn}^{\circ}\eta_{jk}\eta_{np} = (\varepsilon_{ij}^{\circ}\eta_{ij})^2 + \varepsilon_{ij}^{\circ}\varepsilon_{in}^{\circ}\eta_{kj}\eta_{kn} + \varepsilon_{ij}^{\circ}\varepsilon_{mn}^{\circ}\eta_{jm}\eta_{in} \\ \delta_{ijkmnp}\eta_{km}\eta_{rp}\varepsilon_{ij}^{\circ}\varepsilon_{nr}^{\circ} = 2(\varepsilon_{ij}^{\circ}\eta_{ij})^2 + 4\varepsilon_{ik}^{\circ}\varepsilon_{im}^{\circ}\eta_{jk}\eta_{jm} + 4\varepsilon_{ij}^{\circ}\varepsilon_{nr}^{\circ}\eta_{ir}\eta_{jn} \\ \delta_{ijkmnpqr}\varepsilon_{ij}^{\circ}\varepsilon_{pn}^{\circ}\eta_{km}\eta_{qr} = 4\varepsilon_{ij}^{\circ}\varepsilon_{ij}^{\circ}\varepsilon\eta_{km}\eta_{km} + 8(\varepsilon_{ij}^{\circ}\eta_{ij})^2 + 32\varepsilon_{ik}^{\circ}\varepsilon_{im}^{\circ}\eta_{nk}\eta_{nm} + \\ + 16\varepsilon_{ij}^{\circ}\varepsilon_{mn}^{\circ}\eta_{in}\eta_{jm}\end{gathered} \qquad (2.13)$$

3. Let us evaluate the moments of the random fluctuations in (1. 4). The integral (2. 7) is evaluated by means of (2. 8), (2. 10)–(2. 12)

$$\langle k'\varepsilon_{ij}'\rangle = -\frac{2d^2}{5\eta_0}\eta_{ij} \qquad (3.1)$$

Utilizing the identities (2. 13), the moments $\langle\varepsilon'_{ij}\varepsilon'_{ij}\rangle$, $\overset{\circ}{\varepsilon}_{ij}\overset{\circ}{\varepsilon}_{mn} < \langle\varepsilon'_{ij}\varepsilon'_{mn}\rangle$ are analogously evaluated. We finally obtain the dissipative function (1. 4) in the form

$$D = k^{\circ}J_0 - \frac{2d^2\varepsilon_{ij}^{\circ}\nu_{ij}}{5J_0} + \frac{d^2k^{\circ}}{J_0}\{{}^1/_5\nu_{ij}\nu_{ij} - {}^1/_{945}[8\nu_{ij}\nu_{ij} + 70(c_{ij}\nu_{ij})^2 + 46c_{im}c_{ik}\nu_{jm}\nu_{jk} + $$
$$+ 10\varepsilon_{ij}^{\circ}\varepsilon_{mn}^{\circ}\nu_{in}\nu_{jm}]\} + \cdots \qquad (3.2)$$
$$(c_{ij} = \varepsilon_{ij}^{\circ}/J_0,\ \nu_{ij} = \eta_{ij}/\eta_0)$$

Let us assume the fluctuations in the plasticity limit to be small. Then to second order accuracy, (3. 2) will be a quadratic form in the variables ν_{ij}. Since $c_{ij} \leqslant 1$, and the numerical coefficients of the components in the square brackets of (3. 2) are small compared with ${}^1/_5$, the expression in the braces is approximately equal to the positive definite quadratic form ${}^1/_5\,\nu_{ij}\nu_{ij}$.

Taking account of this estimate, we write

$$D = k^{\circ}J_0 + \frac{d^2(k^{\circ}\nu_{ij}\nu_{ij} - 2\varepsilon_{ij}\nu_{ij})}{5J_0} \qquad (3.3)$$

The minimum value of D will be

$$D = \left(k^\circ - \frac{d^2}{5k^\circ}\right) \sqrt{\varepsilon_{ij}{}^\circ \varepsilon_{ij}{}^\circ} \tag{3.4}$$

The dissipative function (3, 3) corresponds to the Mises plasticity condition. The plasticity limit can be expressed by means of given concentrations and plasticity limits of each component. In the case of a two-component medium the plasticity limit is calculated by means of the formula

$$k = c_1k_1 + c_2k_2 - \frac{c_1c_2\,(k_1 - k_2)^2}{5\,(c_1k_1 + c_2k_2)} \tag{3.5}$$

where c_1, c_2, k_1, k_2 are the concentrations and plasticity limits of the corresponding components.

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EXTENDED ORTHOGONALITY RELATIONSHIPS FOR SOME PROBLEMS OF ELASTICITY THEORY

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Orthogonality relationships are derived for the extended eigenvectors of problems on the deformation of a strip, a circular rectangle and the axisymmetric deformation of a cylinder under homogeneous boundary conditions in the displacements.

The problem of the simultaneous decomposition of the boundary conditions given on parts of the surface of an elastic body into a series of nonorthogonal homogeneous solutions is solved only for certain classical problems for definite combinations of the boundary conditions. In the case of the plane problem of the theory of elasticity for a strip, such decompositions are realizable because of the generalized orthogonality relationship of Papkovich [1−4]. A similar relationship for the axisymmetric problem of a cylinder is obtained in [5] and generalized in [6]. However, the mentioned orthogonality relationships do not allow satisfaction of arbitrary boundary conditions exactly on all surfaces of an elastic body of finite size.

Of interest in this respect are the orthogonality relationships of extended eigenvectors of boundary value problems. The elasticity theory equations admit the non-unique construction of such vectors. Thus, Little and Childs [7, 8] construct a system of extended eigenvectors which are orthogonal to the vectors of the conjugate problems. The authors called such orthogonality relationships biorthogonality.

The method developped in [9], which permits construction of a system of extended